

A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes

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(Dated: February 1, 2008)

Abstract

Recent progress in unitarity techniques for one-loop scattering amplitudes makes a numerical implementation of this method possible. We present a 4-dimensional unitarity method for calculating the cut-constructible part of amplitudes and implement the method in a numerical procedure. Our technique can be applied to any one-loop scattering amplitude and offers the possibility that one-loop calculations can be performed in an automatic fashion, as tree-level amplitudes are currently done. Instead of individual Feynman diagrams, the ingredients for our one-loop evaluation are tree-level amplitudes, which are often already known. To study the practicality of this method we evaluate the cut-constructible part of the 4, 5 and 6 gluon one-loop amplitudes numerically, using the analytically known 4, 5 and 6 gluon tree-level amplitudes. Comparisons with analytic answers are performed to ascertain the numerical accuracy of the method.

PACS numbers: 13.85.-t, 13.85.Qk

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I. INTRODUCTION

Analytic unitarity techniques in Feynman diagram calculations have been used for a long time [1, 2, 3]. Their use in the context of gauge theories is even more powerful [4, 5] and they were successfully applied to the calculation of one-loop amplitudes of phenomenologically important 5-leg and 6-leg processes in QCD [6] (for a recent review see [7]).

In gauge theories the conventional Feynman diagram method produces intermediate results which are much more complicated than the final answer. One evaluates the numerous non gauge-invariant individual Feynman diagrams by expanding the tensor loop integrals into form factors. This decomposition generates a large number of terms. With a growing number of external particles it becomes a forbidding task to simplify the expression analytically. This forces one to adopt more numerical techniques (see e.g. ref. [8]), which can be computationally intensive due to the large number of terms. In addition, the large cancellations between the Feynman diagrams can potentially lead to numerical instabilities.

As an alternative, the unitarity cut method uses only on-shell states, manipulates gauge invariant amplitudes and has been used to derive simple answers with simple intermediate steps [6]. New ideas on twistors [9], multipole cuts (generalized unitarity) [10], recursion relations [11, 12, 13, 14], algebraic reduction of tensor integrals [15] and unitarity in D -dimension [16, 17, 18, 19] have made the unitarity cut method even more promising. It appears that ultimately one can find an efficient algorithm which can be used to calculate the one-loop amplitudes in terms of tree-level amplitudes. The progress is due to three important observations.

First, any one-loop amplitude can be decomposed in terms of scalar box, triangle and bubble master integrals where both internal and external particles can be massive or off-shell [20, 21]. The master integrals¹ have to be calculated in dimensional regularization and may have infrared (boxes and triangles) or ultraviolet divergences (bubbles).

The second key observation concerns the application of unitarity techniques to amplitudes with multiple cuts. Using unitarity techniques the coefficients of the master integrals are determined by multiple cuts of the amplitude which place the cut internal lines on their mass shell. After cutting, the tree-level 3-gluon scattering amplitude with all 3 gluons on-shell

¹ For a collection of currently known one-loop master integrals, see the web-site <http://qcdloop.fnal.gov>.

can appear at a vertex of the diagram. These 3-particle amplitudes are identically zero by momentum conservation. Therefore coefficients of the associated master integral can not be extracted. This obstacle is removed by the observation that the tree-level helicity amplitudes can be analytically continued to complex momentum values [9, 10] allowing for solutions to the unitarity constraints in terms of the complex loop momentum. All of the relevant tree-level expressions are non-zero and the appropriate one-loop amplitude is reconstructible from the tree-level amplitudes. With this method the coefficients of the 4-point master integrals can be extracted both analytically and numerically. They are given in terms of the product of 4 tree-level amplitudes, evaluated with the complex on-shell loop momenta [10]. However the coefficients of the 3- and 2-point master integrals were still difficult to extract because terms already included in the 4-point contributions had to be subtracted. It was not clear how to express this subtraction in terms of the corresponding tree-level amplitudes.

The third important observation is that there is a systematic way [15] of calculating the subtraction terms at the integrand level. By manipulating the one-loop amplitude before the loop integration is carried out, the unitarity method is reduced to the algebraic problem of a multi-pole expansion of a rational function. Alternatively, the method of ref. [15] can also be viewed as the calculation of the residues of each pole term of the integrand. The resulting 4-propagator pole (i.e. box contribution), 3-propagator pole (triangle contribution) and 2-propagator pole (bubble contribution) naturally decompose the loop momentum integration vector into a “physical” space spanned by the respective external momenta and the remaining “trivial” space orthogonal to the “physical” space. The so-called spurious terms (or subtraction terms) of ref. [15] are determined by the most generally allowed dependence of the residue on the components of the loop momentum in the “trivial” space. By definition, these spurious terms vanish upon integration over the loop momentum. These ideas allow the extraction of all the coefficients of the master integrals for a given one-loop amplitude. A possible algorithm for analytical extraction of the coefficients within the unitarity method has been worked out in ref. [22].

In this paper we will expand on the algebraic method [15] by developing a numerical scheme. With the numerical method outlined in this paper we evaluate only the cut-constructible part of the amplitude. We will show that the master integral coefficients are calculated in terms of tree-level amplitudes. This makes it possible to “upgrade” existing leading order generators to produce the cut-constructible part of the one-loop amplitudes.

The “upgrade” requires allowing two of the external momenta in the tree-level amplitude to be complex 4-vectors, while leaving the analytic expression of the tree-level amplitude unchanged. For example, to evaluate the cut-constructible part of the 6-gluon amplitude only the analytic 3-, 4-, 5- and 6-gluon tree-level amplitudes are needed (expressed in spinor product language or any other form). Alternatively, one could use an efficient recursive numerical method to evaluate the tree-level amplitude [11, 12, 23].

Because we implement 4-dimensional unitarity cuts, the so-called rational part is not generated [4, 7]. In principle we could expand our scheme to D -dimensional unitarity cuts, thereby generating the complete amplitude. However, it is not immediately clear how to do this while maintaining the requirement of 4-dimensional tree-level building blocks. Alternative methods exist to determine the rational part, which in principle can be combined with the numerical method outlined in this paper to give the complete scattering amplitude. A method for determining the rational part using on-shell recursion relation has been successfully developed and used [24, 25]. A direct numerical implementation of this method should be possible and especially attractive due to the recursive nature of this method. Direct numerical implementation of the D -dimensional unitarity method [16, 17, 18, 19] is harder but may also become practical in the near future. Other methods have been developed using Feynman diagram expansions. While these methods are attractive as far as simplicity goes, they re-introduce all the problems of standard Feynman diagram calculations. In ref. [26] a method for calculating the rational part is developed partially based on Feynman diagram calculations. In refs. [27, 28] a method is proposed to use simplified Feynman diagram techniques for calculating the rational part.

In section 2 we will derive the formalism, which we then apply in section 3 to calculate the cut-constructible part of the 4-, 5- and 6-gluon amplitudes at one-loop by only using the tree-level amplitudes for the 4-, 5- and 6-gluon amplitudes. We will then compare the numerical results to the analytic calculations.

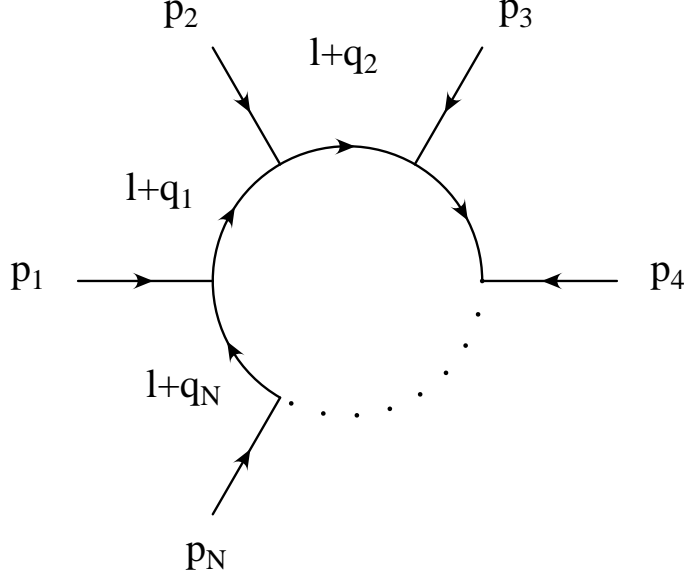


FIG. 1: The generic N -point loop amplitude.

II. THE STRUCTURE OF THE ONE-LOOP INTEGRAND FUNCTION

The generic D -dimensional N -particle one-loop amplitude (fig. 1) is given by²

$$\mathcal{A}_N(p_1, p_2, \dots, p_N) = \int [dl] \frac{\mathcal{N}(p_1, p_2, \dots, p_N; l)}{d_1 d_2 \cdots d_N}, \quad (1)$$

where p_i represent the momenta flowing into the amplitude, and $[dl] = d^D l$. The numerator structure $\mathcal{N}(p_1, p_2, \dots, p_N; l)$ is generated by the particle content and is a function of the inflow momenta and the loop momentum. Since the whole amplitude has been put on a common denominator, the numerator can also include some propagator factors. The dependence of the amplitude on other quantum numbers has been suppressed. The denominator is a product of inverse propagators

$$d_i = d_i(l) = (l + q_i)^2 - m_i^2 = \left(l - q_0 + \sum_{j=1}^i p_j \right)^2 - m_i^2, \quad (2)$$

where the 4-vector q_0 represents the arbitrary parameterization choice of loop momentum. The one-loop amplitude in $D = 4 - 2\epsilon$ can be decomposed in the scalar master integral basis

² We restrict our discussion to (color) ordered external legs. The extension for more general cases is straightforward.

giving ³

$$\begin{aligned}
\mathcal{A}_N(p_1, p_2, \dots, p_N) = & \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq N} d_{i_1 i_2 i_3 i_4}(p_1, p_2, \dots, p_N) I_{i_1 i_2 i_3 i_4} \\
& + \sum_{1 \leq i_1 < i_2 < i_3 \leq N} c_{i_1 i_2 i_3}(p_1, p_2, \dots, p_N) I_{i_1 i_2 i_3} \\
& + \sum_{1 \leq i_1 < i_2 \leq N} b_{i_1 i_2}(p_1, p_2, \dots, p_N) I_{i_1 i_2} \\
& + \sum_{1 \leq i_1 \leq N} a_{i_1}(p_1, p_2, \dots, p_N) I_{i_1} ,
\end{aligned} \tag{3}$$

where the master integrals are given by

$$I_{i_1 \dots i_M} = \int [d l] \frac{1}{d_{i_1} \dots d_{i_M}} . \tag{4}$$

Analytic expressions for the master integrals with massless internal lines are reported in ref. [5].

The maximum number of master integrals is determined by the dimensionality, D , of space-time; for the physical case this gives up to 4-point master integrals. The unitarity cut method is based on the study of the analytic structure of the one-loop amplitude. The coefficients are rational functions of the kinematical variables and will in general depend on the dimensional regulator variable $\epsilon = (4 - D)/2$. When all the coefficients of the master integrals are calculated in 4 dimensions we obtain the “cut-constructible” part of the amplitude. The remaining “rational part” is generated by the omitted $\mathcal{O}(\epsilon)$ part of the master integral coefficients [7].

For a numerical procedure we need to recast the study of the analytic properties of the unitarity cut amplitudes into an algebraic algorithm which can be implemented numerically. In ref. [15] it was proposed that one focus on the integrand of the one-loop amplitude,

$$\mathcal{A}_N(p_1, p_2, \dots, p_N | l) = \frac{\mathcal{N}(p_1, p_2, \dots, p_N; l)}{d_1 d_2 \dots d_N} . \tag{5}$$

This is a rational function of the loop momentum. Any N -point tensor integral of rank M ($M \leq N$) with $N \geq 5$ can be reduced to 4-point tensor integrals of rank K ($K \leq 4$) by application of Schouten identities. Therefore we can re-express the rational function in an

³ We drop the finite 6-dimensional 5-point master integral because its coefficient is of $\mathcal{O}(\epsilon)$ [29] and therefore it will not contribute to the final answer where we take $\epsilon \rightarrow 0$.

expansion over 4-, 3-, 2- and 1-propagator pole terms. The residues of these pole terms contain the master integral coefficients as well as structures which reside in the subspace orthogonal to the subspace spanned by the external momenta. These spurious terms are important as subtraction terms in the determination of lower multiplicity poles. The number of spurious structures is 1 for the box, 8 for the triangle, 6 for the bubble, and 4 for the tadpole. After integration over the loop momenta, Eq. (3) is recovered. This approach transforms the analytic unitarity method into the algebraic problem of partial fractioning a multi-pole rational function. The remaining integrals after the partial fractioning are guaranteed to be the master integrals of Eq. (4). This makes a numerical implementation feasible.

A. The van Neerven-Vermaseren basis

Consider a set of R inflow momenta, k_1, \dots, k_R in a D -dimensional space-time⁴. Taking momentum conservation into account, $\sum_{i=1}^R k_i = 0$, the physical space spanned by the momenta k_i has dimension $\min(D, R - 1)$. As a consequence, for $R \geq D + 1$ additional Schouten identities exist, which can be exploited to prove the $D = 4$ master integral basis of Eq. (3) [21]. For $R \leq D$, the physical space forms a lower dimensional subspace. We can define an orthonormal basis, the van Neerven-Vermaseren (NV) basis [30], which separates the D -dimensional space into the D_P -dimensional “physical” space and the orthogonal D_T -dimensional “trivial” space where

$$D = D_P + D_T; \quad D_P = \min(D, R - 1); \quad D_T = \max(0, D - R + 1) . \quad (6)$$

⁴ The inflow momenta are either equal to the external momenta p_i , or to sums of external momenta.

To define the NV-basis we introduce the generalized Kronecker delta [31]⁵

$$\delta_{\nu_1 \nu_2 \dots \nu_R}^{\mu_1 \mu_2 \dots \mu_R} = \begin{vmatrix} \delta_{\nu_1}^{\mu_1} & \delta_{\nu_2}^{\mu_1} & \dots & \delta_{\nu_R}^{\mu_1} \\ \delta_{\nu_1}^{\mu_2} & \delta_{\nu_2}^{\mu_2} & \dots & \delta_{\nu_R}^{\mu_2} \\ \vdots & \vdots & & \vdots \\ \delta_{\nu_1}^{\mu_R} & \delta_{\nu_2}^{\mu_R} & \dots & \delta_{\nu_R}^{\mu_R} \end{vmatrix}, \quad (7)$$

the compact notation

$$\delta_{\nu_1 q \dots \nu_R}^{\mu_1 \mu_2 \dots \mu_R} \equiv \delta_{\nu_1 \nu_2 \dots \nu_R}^{\mu_1 \mu_2 \dots \mu_R} k_{\mu_1} q^{\nu_2}, \quad (8)$$

and the $(R-1)$ -particle Gram determinant

$$\Delta(k_1, k_2, \dots, k_{R-1}) = \delta_{k_1 k_2 \dots k_{R-1}}^{k_1 k_2 \dots k_{R-1}}. \quad (9)$$

Note that for $R \geq D+1$ the generalized Kronecker delta is zero. For the special case $D = R$ we have the factorization of the Kronecker delta into a product of Levi-Civita tensors: $\delta_{\nu_1 \nu_2 \dots \nu_R}^{\mu_1 \mu_2 \dots \mu_R} = \varepsilon^{\mu_1 \mu_2 \dots \mu_R} \varepsilon_{\nu_1 \nu_2 \dots \nu_R}$.

Some examples of the generalized Kronecker delta are

$$\begin{aligned} \delta_{q_1 \mu}^{k_1 k_2} &= k_1 \cdot q_1 \delta_{\mu}^{k_2} - k_{1\mu} \delta_{q_1}^{k_2} \\ &= k_1 \cdot q_1 k_{2\mu} - k_2 \cdot q_1 k_{1\mu} \\ \delta_{q_1 q_2 q_3}^{k_1 k_2 k_3} &= k_1 \cdot q_1 \delta_{q_2 q_3}^{k_2 k_3} - k_1 \cdot q_2 \delta_{q_1 q_3}^{k_2 k_3} + k_1 \cdot q_3 \delta_{q_1 q_2}^{k_2 k_3} \\ &= +k_1 \cdot q_1 (k_2 \cdot q_2 k_3 \cdot q_3 - k_2 \cdot q_3 k_3 \cdot q_2) \\ &\quad - k_1 \cdot q_2 (k_2 \cdot q_1 k_3 \cdot q_3 - k_2 \cdot q_3 k_3 \cdot q_1) \\ &\quad + k_1 \cdot q_3 (k_2 \cdot q_1 k_3 \cdot q_2 - k_2 \cdot q_2 k_3 \cdot q_1). \end{aligned} \quad (10)$$

We now want to construct the NV-basis for R momenta. We define D_P basis vectors

$$v_i^\mu(k_1, \dots, k_{D_P}) \equiv \frac{\delta_{k_1 \dots k_{i-1} k_i k_{i+1} \dots k_{D_P}}^{k_1 \dots k_{i-1} \mu k_{i+1} \dots k_{D_P}}}{\Delta(k_1, \dots, k_{D_P})}, \quad (11)$$

⁵ This notation is closely related to the asymmetric Gram determinant notation of ref. [32],

$$G \begin{pmatrix} k_1 & \dots & k_R \\ q_1 & \dots & q_R \end{pmatrix} = \delta_{q_1 q_2 \dots q_R}^{k_1 k_2 \dots k_R}.$$

with the properties $v_i \cdot k_j = \delta_{ij}$ for $j \leq D_P$. When $R \leq D$ we also need to define the projection operator onto the trivial space

$$w_\mu{}^\nu(k_1 \dots k_{R-1}) \equiv \frac{\delta_{k_1 \dots k_{R-1} \mu}^{k_1 \dots k_{R-1} \nu}}{\Delta(k_1, \dots, k_{R-1})} , \quad (12)$$

with the properties $w_\mu{}^\mu = D_T = D + 1 - R$, $k_i^\mu w_{\mu\nu} = 0$ and $w^\mu{}_\alpha w^{\alpha\nu} = w^{\mu\nu}$. Note that this operator is the metric tensor of the trivial subspace, with the decomposition

$$w^{\mu\nu} = \sum_{i=1}^{D+1-R} n_i^\mu n_i^\nu , \quad (13)$$

where the $D + 1 - R$ orthonormal base vectors of the trivial space n_i have the property $n_i \cdot n_j = \delta_{ij}$, $n_i \cdot k_j = n_i \cdot v_j = 0$.

The full metric tensor decomposition in the NV-basis is given by ⁶

$$g^{\mu\nu} = \sum_{i=1}^{D_P} k_i^\mu v_i^\nu + w^{\mu\nu} = \sum_{i=1}^{D_P} k_i^\mu v_i^\nu + \sum_{i=1}^{D_T} n_i^\mu n_i^\nu . \quad (14)$$

For the case $D = R$ the sole basis vector of the 1-dimensional trivial space is proportional to the Levi-Civita tensor. For the cases $R < D$ we can explicitly construct the basis vectors fulfilling all the requirements.

As an example in the case of $D = 4$ and $R = 4$ we get

$$\begin{aligned} v_1^\mu(k_1, k_2, k_3) &= \frac{\delta_{k_1 k_2 k_3}^{\mu k_2 k_3}}{\Delta(k_1, k_2, k_3)}; \quad v_2^\mu(k_1, k_2, k_3) = \frac{\delta_{k_1 k_2 k_3}^{k_1 \mu k_3}}{\Delta(k_1, k_2, k_3)}; \quad v_3^\mu(k_1, k_2, k_3) = \frac{\delta_{k_1 k_2 k_3}^{k_1 k_2 \mu}}{\Delta(k_1, k_2, k_3)} \\ w_\mu{}^\nu(k_1, k_2, k_3) &= \frac{\delta_{k_1 k_2 k_3 \mu}^{k_1 k_2 k_3 \nu}}{\Delta(k_1, k_2, k_3)} = n_{1\mu} n_1^\nu = \frac{\varepsilon_{k_1 k_2 k_3 \mu} \varepsilon^{k_1 k_2 k_3 \nu}}{\Delta(k_1, k_2, k_3)} . \end{aligned} \quad (15)$$

We want to decompose the loop momentum into the NV-basis for a graph with denominator factor d_1, d_2, \dots, d_R . The denominators are as usual given by $d_i = (l + q_i)^2 - m_i^2$ and $k_i = q_i - q_{i-1}$. By contracting in the loop momentum with the metric tensor given in Eq. (14) we get the loop momentum decomposition in the NV-basis

$$l^\mu = \sum_{i=1}^{D_P} l \cdot k_i v_i^\mu + \sum_{i=1}^{D_T} l \cdot n_i n_i^\mu . \quad (16)$$

Using the notation $l \cdot n_i = \alpha_i(l) = \alpha_i$ and the identity

$$l \cdot k_i = \frac{1}{2} [d_i - d_{i-1} - (q_i^2 - m_i^2) + (q_{i-1}^2 - m_{i-1}^2)] , \quad (17)$$

⁶ By expanding the generalized Kronecker delta functions in the v_i vectors one can show that $\sum_i k_i^\mu v_i^\nu = \sum_i k_i^\nu v_i^\mu$.

we find

$$l^\mu = V_R^\mu + \sum_{i=1}^{D_P} \frac{1}{2} (d_i - d_{i-1}) v_i^\mu + \sum_{i=1}^{D_T} \alpha_i n_i^\mu, \quad (18)$$

where $d_0 = d_R$, $m_0 = m_R$ and

$$V_R^\mu = -\frac{1}{2} \sum_{i=1}^{D_P} \left((q_i^2 - m_i^2) - (q_{i-1}^2 - m_{i-1}^2) \right) v_i^\mu. \quad (19)$$

In the case that $R \geq D + 1$ the decomposition of the loop momentum into the NV-basis implicitly proves Eq. (3). Also, when $R \leq D$ it allows us to include the unitarity constraints without resorting to the explicit 4-dimensional spinor formalism used in analytic calculations. By avoiding the 4-dimensional spinor formalism, the formulation is also valid for massive internal particles (where the mass can be real or complex valued).

For example, in the case of a 4-dimensional pentagon, ($D = 4$ and $R = 5$) we get

$$\begin{aligned} l^\mu &= V_5^\mu + \frac{1}{2} (d_1 - d_5) v_1^\mu + \frac{1}{2} (d_2 - d_1) v_2^\mu + \frac{1}{2} (d_3 - d_2) v_3^\mu + \frac{1}{2} (d_4 - d_3) v_4^\mu \\ V_5^\mu &= -\frac{1}{2} (q_1^2 - q_5^2 - m_1^2 + m_5^2) v_1^\mu - \frac{1}{2} (q_2^2 - q_1^2 - m_2^2 + m_1^2) v_2^\mu \\ &\quad - \frac{1}{2} (q_3^2 - q_2^2 - m_3^2 + m_2^2) v_3^\mu - \frac{1}{2} (q_4^2 - q_3^2 - m_4^2 + m_3^2) v_4^\mu. \end{aligned} \quad (20)$$

Similarly for a 4-dimensional triangle ($D = 4$ and $R = 3$) we get

$$\begin{aligned} l^\mu &= V_3^\mu + \frac{1}{2} (d_1 - d_3) v_1^\mu + \frac{1}{2} (d_2 - d_1) v_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu \\ V_3^\mu &= -\frac{1}{2} (q_1^2 - q_3^2 - m_1^2 + m_3^2) v_1^\mu - \frac{1}{2} (q_2^2 - q_1^2 - m_2^2 + m_1^2) v_2^\mu. \end{aligned} \quad (21)$$

Thus we see that the same basis decomposition is used for the tensor reductions in the case $R \geq D + 1$ and solving the unitarity constraint in the case that $R \leq D$.

B. Partial fractioning of the integrand

For the remainder of the paper we restrict ourselves to a 4-dimensional space. Given the master integral decomposition of Eq. (3) we can partial fraction the integrand of any 4-dimensional N -particle amplitude as

$$\mathcal{A}_N(l) = \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq N} \frac{\bar{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \leq i_1 < i_2 < i_3 \leq N} \frac{\bar{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \leq i_1 < i_2 \leq N} \frac{\bar{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \leq i_1 \leq N} \frac{\bar{a}_{i_1}(l)}{d_{i_1}}.$$

To calculate the numerator factors, we will calculate the residues by taking the inverse propagators equal to zero. The residue has to be taken by constructing the loop momentum

$l_{ij\dots k}$ such that $d_i(l_{ij\dots k}) = d_j(l_{ij\dots k}) = \dots = d_k(l_{ij\dots k}) = 0$. Then the residue of a function $F(l)$ is given by

$$\text{Res}_{ij\dots k} [F(l)] \equiv \left(d_i(l) d_j(l) \dots d_k(l) F(l) \right) \Big|_{l=l_{ij\dots k}} . \quad (22)$$

The specific residues are now given by

$$\begin{aligned} \bar{d}_{ijkl}(l) &= \text{Res}_{ijkl} \left(\mathcal{A}_N(l) \right) \\ \bar{c}_{ijk}(l) &= \text{Res}_{ijk} \left(\mathcal{A}_N(l) - \sum_{l \neq i,j,k} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right) \\ \bar{b}_{ij}(l) &= \text{Res}_{ij} \left(\mathcal{A}_N(l) - \sum_{k \neq i,j} \frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{2!} \sum_{k,l \neq i,j} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right) \\ \bar{a}_i(l) &= \text{Res}_i \left(\mathcal{A}_N(l) - \sum_{j \neq i} \frac{\bar{b}_{ij}(l)}{d_i d_j} - \frac{1}{2!} \sum_{j,k \neq i} \frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{3!} \sum_{j,k,l \neq i} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right) . \end{aligned} \quad (23)$$

Note that the coefficients are defined to be symmetric in the propagator indices (e.g. $\bar{c}_{123} = \bar{c}_{213} = \bar{c}_{312}$) and coefficients with repeated indices are to be set to zero (e.g. $d_{1336} = c_{112} = 0$).

As an example, some residues of a 5-particle amplitude are given by

$$\begin{aligned} \bar{d}_{1245} &= \text{Res}_{1245} \left(\mathcal{A}_N(l) \right) \\ \bar{c}_{235} &= \text{Res}_{235} \left(\mathcal{A}_N(l) - \frac{\bar{d}_{1235}(l)}{d_1 d_2 d_3 d_5} - \frac{\bar{d}_{2345}(l)}{d_2 d_3 d_4 d_5} \right) \\ \bar{b}_{14} &= \text{Res}_{14} \left(\mathcal{A}_N(l) - \frac{\bar{c}_{124}(l)}{d_1 d_2 d_4} - \frac{\bar{c}_{134}(l)}{d_1 d_3 d_4} - \frac{\bar{c}_{145}(l)}{d_1 d_4 d_5} - \frac{\bar{d}_{1234}(l)}{d_1 d_2 d_3 d_4} - \frac{\bar{d}_{1245}(l)}{d_1 d_2 d_4 d_5} - \frac{\bar{d}_{1345}(l)}{d_1 d_3 d_4 d_5} \right) . \end{aligned} \quad (24)$$

In the following sub-sections we will explicitly construct the residue functions using only tree-level amplitudes. This construction is well-suited for numerical implementation.

C. Constructing the box residue

To calculate the box coefficients we choose the loop momentum l_{ijkl} such that four inverse propagators are equal to zero,

$$\bar{d}_{ijkl}(l_{ijkl}) = \text{Res}_{ijkl} \left(\mathcal{A}_N(l) \right) . \quad (25)$$

We will drop the subscripts on the loop momentum in the following. Because we have to solve the unitarity constraints explicitly, we have to choose a specific parameterization, q_0 , in Eq. (2). Using the NV-basis of the four inflow momenta for the box and using the fact that $d_i = d_j = d_k = d_l = 0$ we can use Eq. (18) to decompose the loop momentum as

$$l^\mu = V_4^\mu + \alpha_1 n_1^\mu . \quad (26)$$

Choosing for the parameterization $q_0 = \sum_{j=1}^l p_j$ (as usual the index i is understood to be modulo N) such that $q_l = 0$ we have

$$V_4^\mu = -\frac{1}{2}(q_i^2 - m_i^2 + m_l^2) v_1^\mu - \frac{1}{2}(q_j^2 - q_i^2 - m_j^2 + m_i^2) v_2^\mu - \frac{1}{2}(q_k^2 - q_j^2 - m_k^2 + m_j^2) v_3^\mu , \quad (27)$$

where

$$v_1^\mu = \frac{\delta_{k_1 k_2 k_3}^{\mu k_2 k_3}}{\Delta(k_1, k_2, k_3)}; \quad v_2^\mu = \frac{\delta_{k_1 k_2 k_3}^{k_1 \mu k_3}}{\Delta(k_1, k_2, k_3)}; \quad v_3^\mu = \frac{\delta_{k_1 k_2 k_3}^{k_1 k_2 \mu}}{\Delta(k_1, k_2, k_3)}; \quad n_1^\mu = \frac{\varepsilon^{\mu k_1 k_2 k_3}}{\sqrt{\Delta(k_1, k_2, k_3)}} , \quad (28)$$

and

$$k_1 = q_i; \quad k_2 = q_j - q_i; \quad k_3 = q_k - q_j; \quad \Delta(k_1, k_2, k_3) = \delta_{k_1 k_2 k_3}^{k_1 k_2 k_3} . \quad (29)$$

The variable α_1 will be determined such that the unitarity condition $d_i = d_j = d_k = d_l = 0$ is fulfilled⁷. Imposing the constraint $d_n = 0$ for $n = \{i, j, k, l\}$ using Eq. (26)

$$\begin{aligned} d_n = 0 &= (l + q_n)^2 - m_n^2 \\ &= (V_4 + \alpha_1 n_1 + q_n)^2 - m_n^2 \\ &= V_4^2 + \alpha_1^2 + 2V_4 \cdot q_n + q_n^2 - m_n^2 , \end{aligned} \quad (30)$$

and using the identity

$$V_4 \cdot q_n = -\frac{1}{2}(q_n^2 - m_n^2 + m_l^2) , \quad (31)$$

so that

$$\alpha_1^2 = -(V_4^2 - m_l^2) , \quad (32)$$

we find two complex solutions

$$l_\pm^\mu = V_4^\mu \pm i \sqrt{V_4^2 - m_l^2} \times n_1^\mu , \quad (33)$$

⁷ In fact $\alpha_1 = \alpha_1(l)$, by varying the loop momentum (allowing for complex values) we can set α_1 to the required complex value. This dependence is implicitly assumed, we will treat α_1 as a “free variable”.

which are easily numerically implemented. We note that because $d_i = d_j = d_k = d_l = 0$, we have $(l_n^\pm)^2 = m_n^2$ for $n = i, j, k, l$ where $l_n^\pm = l_{ijkl}^\pm + q_n = l_{ijkl}^\pm + \sum_{j=l+1}^n p_i$. In other words the four propagators are on-shell and the amplitude factorizes for a given intermediate state into 4 tree-level amplitudes $\mathcal{M}^{(0)}$. For the residue of the amplitude in Eq. (25) we find (all indices are assumed modulo N , i.e. $i = n + N = n$)

$$\begin{aligned} \text{Res}_{ijkl}(\mathcal{A}_N(l^\pm)) &= \mathcal{M}^{(0)}(l_i^\pm; p_{i+1}, \dots, p_j; -l_j^\pm) \times \mathcal{M}^{(0)}(l_j^\pm; p_{j+1}, \dots, p_k; -l_k^\pm) \\ &\times \mathcal{M}^{(0)}(l_k^\pm; p_{k+1}, \dots, p_l; -l_l^\pm) \times \mathcal{M}^{(0)}(l_l^\pm; p_{l+1}, \dots, p_i; -l_i^\pm), \end{aligned} \quad (34)$$

where the loop momenta l_n^μ are complex on-shell momenta and there is an implicit sum over all states of the cut lines (such as e.g. particle type, color, helicity). For example, the residue of the amplitude for the pure 6-gluon ordered amplitude with $d_6 = d_2 = d_3 = d_4 = 0$ factorizes into (see fig. 2)

$$\begin{aligned} \text{Res}_{2346}(\mathcal{A}_6(l^\pm)) &= \mathcal{M}_4^{(0)}(l_6^\pm; p_1, p_2; -l_2^\pm) \times \mathcal{M}_3^{(0)}(l_2^\pm; p_3; -l_3^\pm) \\ &\times \mathcal{M}_3^{(0)}(l_3^\pm; p_4; -l_4^\pm) \times \mathcal{M}_4^{(0)}(l_4^\pm; p_5, p_6; -l_6^\pm), \end{aligned} \quad (35)$$

where the implicit sum over the two helicity states of the four cut gluons is assumed. The tree-level 3-gluon amplitudes, $\mathcal{M}_3^{(0)}$, are non-zero because the two cut gluons have complex momenta [10].

Any remaining dependence of the residue \bar{d}_{ijkl} on the loop momentum enters through its component in the trivial space,

$$\bar{d}_{ijkl}(l) \equiv \bar{d}_{ijkl}(n_1 \cdot l). \quad (36)$$

The number of powers of the loop momentum l in the numerator structure is called the rank of the integral. After integration we find using Eq. (13) that $(n_1 \cdot l)^2 \sim n_1^2 = 1$. Thus rank one is the maximum rank of a spurious term (which by definition vanishes upon integration over l). Hence the most general form of the residue is

$$\bar{d}_{ijkl}(l) = d_{ijkl} + \tilde{d}_{ijkl} l \cdot n_1. \quad (37)$$

Using the two solutions of the unitarity constraint, Eq. (33), we now can determine the two coefficients of the residue

$$\begin{aligned} d_{ijkl} &= \frac{\text{Res}_{ijkl}(\mathcal{A}_N(l^+)) + \text{Res}_{ijkl}(\mathcal{A}_N(l^-))}{2} \\ \tilde{d}_{ijkl} &= \frac{\text{Res}_{ijkl}(\mathcal{A}_N(l^+)) - \text{Res}_{ijkl}(\mathcal{A}_N(l^-))}{2i\sqrt{V_4^2 - m_l^2}}. \end{aligned} \quad (38)$$

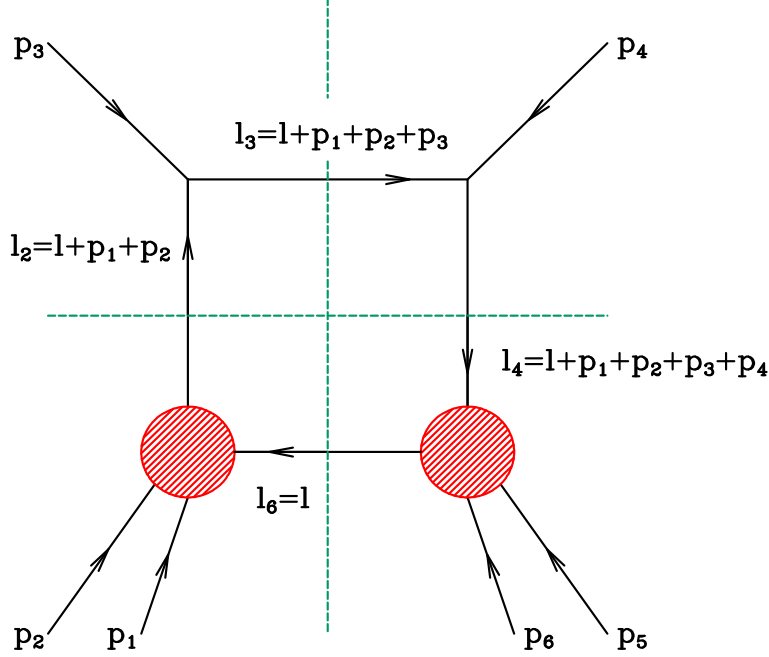


FIG. 2: The factorization of the 6-gluon amplitude for the calculation of the $\bar{d}_{2346}(l)$ residue with the loop momentum parametrization choice $q_0 = 0$.

With the above prescription it is now easy to determine the spurious term for any value of the loop momentum. Finally we note that the integration over the term

$$\int [d l] \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} = \int [d l] \frac{d_{ijkl} + \tilde{d}_{ijkl} n_1 \cdot l}{d_i d_j d_k d_l} = d_{ijkl} \int [d l] \frac{1}{d_i d_j d_k d_l} = d_{ijkl} I_{ijkl} , \quad (39)$$

is now trivially done, giving us the coefficient of the box times the box master integral.

D. Construction of the triangle residue

To calculate the triangle coefficients we need to put three propagators on-shell. Care has to be taken to remove the box contributions by explicit subtraction. Thus, the triangle coefficient is given by

$$\bar{c}_{ijk}(l) = \text{Res}_{ijk} \left(\mathcal{A}_N(l) - \sum_{l \neq i, j, k} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_l} \right) . \quad (40)$$

Decomposing the loop momentum in the NV-basis of the three inflow momenta of the triangle with $d_i = d_j = d_k = 0$ (choosing $q_k = 0$) gives us according to Eq. (18)

$$l^\mu = V_3^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu , \quad (41)$$

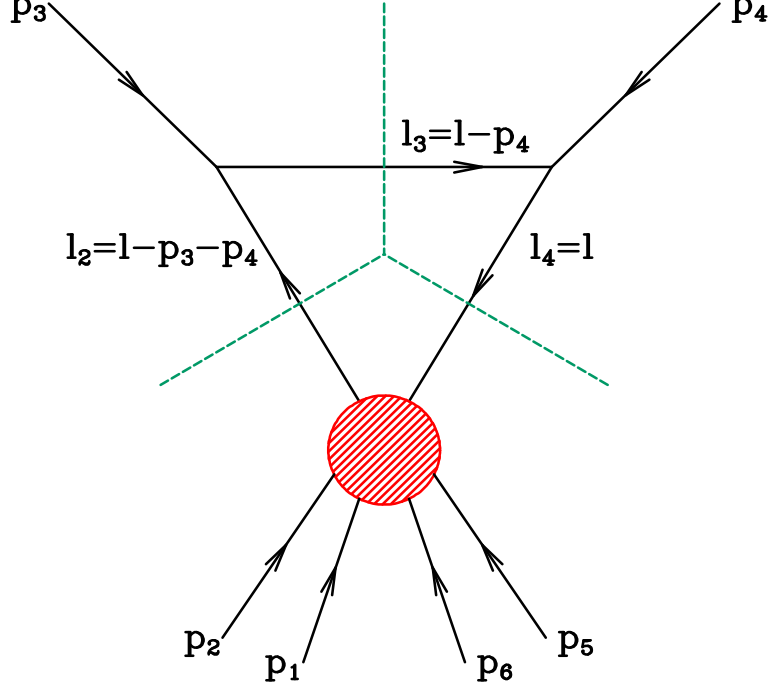


FIG. 3: The factorization of the 6-gluon ordered amplitude for the calculation of the $\bar{c}_{234}(l)$ residue with the loop momentum parametrization choice $q_0 = -p_5 - p_6 = p_1 + p_2 + p_3 + p_4$.

with

$$V_3^\mu = -\frac{1}{2}(q_i^2 - m_i^2 + m_k^2) v_1^\mu - \frac{1}{2}(q_j^2 - q_i^2 - m_j^2 + m_i^2) v_2^\mu, \quad (42)$$

where

$$v_1^\mu = \frac{\delta_{k_1 k_2}^{\mu k_2}}{\Delta(k_1, k_2)}; \quad v_2^\mu = \frac{\delta_{k_1 k_2}^{k_1 \mu}}{\Delta(k_1, k_2)}, \quad (43)$$

and

$$k_1 = q_i; \quad k_2 = q_j - q_i; \quad \Delta(k_1, k_2) = \delta_{k_1 k_2}^{k_1 k_2}. \quad (44)$$

The base vectors of the trivial space $\{n_1, n_2\}$ have to be explicitly constructed using the constraints

$$n_i \cdot n_j = \delta_{ij}; \quad n_i \cdot k_j = 0; \quad w^{\mu\nu}(k_1, k_2, k_3) = n_1^\mu n_1^\nu + n_2^\mu n_2^\nu. \quad (45)$$

The unitarity constraints ($d_i = d_j = d_k = 0$) give an infinite set of solutions ⁸

$$l_{\alpha_1 \alpha_2}^\mu = V_3^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu; \quad \alpha_1^2 + \alpha_2^2 = -(V_3^2 - m_k^2). \quad (46)$$

⁸ For massless internal lines the parameterization of ref. [22] is obtained by taking $\alpha_1 = \kappa \times (a + i b)$ and $\alpha_2 = \kappa \times (a - i b)$ where $\kappa^2 = -V_3^2$, $a = \frac{1}{2}(t - 1/t)$ and $b = \frac{1}{2}(t + 1/t)$. By taking the $t \rightarrow \infty$ limit one gets the coefficient of the triangle master integrals.

In this case the residue of the amplitude factorizes into three tree-level amplitudes

$$\begin{aligned} \text{Res}_{ijk}(\mathcal{A}_N(l^{\alpha_1\alpha_2})) &= \mathcal{M}^{(0)}(l_i^{\alpha_1\alpha_2}; p_{i+1}, \dots, p_j; -l_j^{\alpha_1\alpha_2}) \times \mathcal{M}^{(0)}(l_j^{\alpha_1\alpha_2}; p_{j+1}, \dots, p_k; -l_k^{\alpha_1\alpha_2}) \\ &\times \mathcal{M}^{(0)}(l_k^{\alpha_1\alpha_2}; p_{k+1}, \dots, p_i; -l_i^{\alpha_1\alpha_2}) , \end{aligned} \quad (47)$$

with an implicit sum over the internal states of the cut lines. For example, the residue of the amplitude for the pure 6-gluon amplitude with $d_2 = d_3 = d_4 = 0$ factorizes into (see fig. 3)

$$\begin{aligned} \text{Res}_{234}(\mathcal{A}_6(l^{\alpha_1\alpha_2})) &= \mathcal{M}_3^{(0)}(l_2^{\alpha_1\alpha_2}; p_3; -l_3^{\alpha_1\alpha_2}) \times \mathcal{M}_3^{(0)}(l_3^{\alpha_1\alpha_2}; p_4; -l_4^{\alpha_1\alpha_2}) \\ &\times \mathcal{M}_6^{(0)}(l_4^{\alpha_1\alpha_2}; p_5, p_6, p_1, p_2; -l_2^{\alpha_1\alpha_2}) . \end{aligned} \quad (48)$$

The remaining dependence of the residue \bar{c}_{ijk} on the loop momentum resides in the trivial space

$$\bar{c}_{ijk}(l) \equiv \bar{c}_{ijk}(s_1, s_2); \quad s_1 = n_1 \cdot l, \quad s_2 = n_2 \cdot l . \quad (49)$$

The maximum rank of the triangle diagrams in standard model processes is 3. This gives us 10 possible terms ($\{1, s_1, s_2, s_1^2, s_1s_2, s_2^2, s_1^3, s_1^2s_2, s_1s_2^2, s_2^3\}$) for the most general polynomial form of the residue. However using Eq. (13) we have the constraint $s_1^2 + s_2^2 \sim n_1^2 + n_2^2 = 2$ which reduces the number of terms to 7. The form we chose is given by

$$\bar{c}_{ijk}(l) = c_{ijk}^{(0)} + c_{ijk}^{(1)}s_1 + c_{ijk}^{(2)}s_2 + c_{ijk}^{(3)}(s_1^2 - s_2^2) + s_1s_2(c_{ijk}^{(4)} + c_{ijk}^{(5)}s_1 + c_{ijk}^{(6)}s_2) . \quad (50)$$

We need to determine all 7 constants $c_{ijk}^{(n)}$ by constructing 7 equations. This is accomplished by choosing 7 combinations of (α_1, α_2) with $\alpha_1^2 + \alpha_2^2 = -(V_3^2 - m_k^2)$ in Eq. (40). This system of equations can be easily solved using a matrix inversion. Note that in principle we can generate an unlimited set of equations. This can be useful in a numerical application to obtain better numerical accuracy.

With the above prescription it is now easy to determine the spurious term for any value of the loop momentum. Finally we note that the integration over the term

$$\int [dl] \frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} = c_{ijk}^{(0)} \int [dl] \frac{1}{d_i d_j d_k} = c_{ijk} I_{ijk} , \quad (51)$$

is now trivially done, giving us the triangle coefficient times the triangle master integral.

E. Construction of the bubble residue

This sub-section follows closely the previous two sub-sections. To calculate the bubble coefficients we need to put two propagators on-shell. The box and triangle contributions need to be explicitly subtracted. This gives for the bubble coefficient

$$\bar{b}_{ij}(l) = \text{Res}_{ij} \left(\mathcal{A}_N(l) - \sum_{k \neq i,j} \frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{2!} \sum_{k,l \neq i,j} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right). \quad (52)$$

Decomposing the loop momentum in the NV-basis of the two inflow momenta with $d_i = d_j = 0$ (choosing $q_j = 0$) gives us according to Eq. (18)

$$l^\mu = V_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu, \quad (53)$$

with

$$V_2^\mu = -\frac{1}{2}(q_i^2 - m_i^2 + m_j^2) v_1^\mu, \quad (54)$$

where

$$v_1^\mu = \frac{\delta_{k_1}^\mu}{\Delta(k_1)} = \frac{k_1^\mu}{k_1^2}, \quad (55)$$

and

$$k_1 = q_i. \quad (56)$$

The base vectors of the trivial space $\{n_1, n_2, n_3\}$ have to be explicitly constructed using the constraints

$$n_i \cdot n_j = \delta_{ij}; \quad n_i \cdot k_j = 0; \quad w^{\mu\nu}(k_1, k_2) = n_1^\mu n_1^\nu + n_2^\mu n_2^\nu + n_3^\mu n_3^\nu. \quad (57)$$

The solution to the unitarity constraints ($d_i = d_j = 0$) gives as a infinite set of solutions

$$l_{\alpha_1 \alpha_2 \alpha_3}^\mu = V_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu; \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = -(V_2^2 - m_j^2). \quad (58)$$

As before the residue of the amplitude factorizes into tree-level amplitudes, in this case into 2 tree-level amplitudes with an implicit sum over the states of the cut lines

$$\begin{aligned} \text{Res}_{ij} \left(\mathcal{A}_N(l^{\alpha_1 \alpha_2 \alpha_3}) \right) = \\ \mathcal{M}^{(0)}(l_i^{\alpha_1 \alpha_2 \alpha_3}; p_{i+1}, \dots, p_j; -l_j^{\alpha_1 \alpha_2 \alpha_3}) \times \mathcal{M}^{(0)}(l_j^{\alpha_1 \alpha_2 \alpha_3}; p_{j+1}, \dots, p_k; -l_k^{\alpha_1 \alpha_2 \alpha_3}). \end{aligned} \quad (59)$$

For example, the residue of the amplitude for the pure 6-gluon amplitude with $d_2 = d_4 = 0$ factorizes into (see fig. 4)

$$\text{Res}_{24} \left(\mathcal{A}_6(l^{\alpha_1 \alpha_2 \alpha_3}) \right) = \mathcal{M}_4^{(0)}(l_2^{\alpha_1 \alpha_2 \alpha_3}; p_3, p_4; -l_4^{\alpha_1 \alpha_2 \alpha_3}) \times \mathcal{M}_6^{(0)}(l_4^{\alpha_1 \alpha_2 \alpha_3}; p_5, p_6, p_1, p_2; -l_2^{\alpha_1 \alpha_2 \alpha_3}). \quad (60)$$

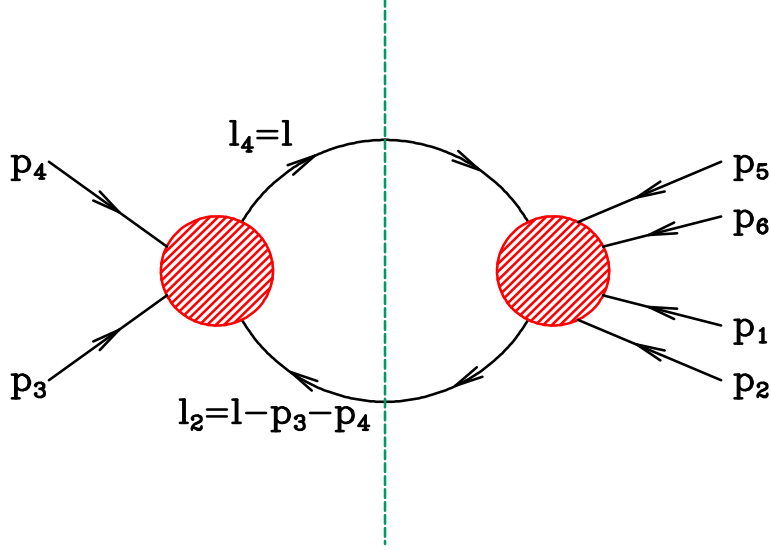


FIG. 4: The factorization of the 6-gluon amplitude for the calculation of the $\bar{b}_{24}(l)$ residue with the loop momentum parametrization choice $q_0 = -p_5 - p_6 = p_1 + p_2 + p_3 + p_4$.

The remaining loop dependence of the residue \bar{b}_{ij} is in the trivial space

$$\bar{b}_{ij}(l) \equiv \bar{b}_{ij}(s_1, s_2, s_3); \quad s_1 = n_1 \cdot l, \quad s_2 = n_2 \cdot l, \quad s_3 = n_3 \cdot l. \quad (61)$$

The maximum rank of the bubble in standard model processes is 2. This gives us 10 possible terms ($\{1, s_1, s_2, s_3, s_1^2, s_2^2, s_3^2, s_1 s_2, s_1 s_3, s_2 s_3\}$) for the most general polynomial form of the residue. However, using Eq. (13) we have the constraint $s_1^2 + s_2^2 + s_3^2 \sim n_1^2 + n_2^2 + n_3^2 = 3$ which reduces the number of terms to 9. The form we chose is given by

$$\bar{b}_{ij}(l) = b_{ij}^{(0)} + b_{ij}^{(1)} s_1 + b_{ij}^{(2)} s_2 + b_{ij}^{(3)} s_3 + b_{ij}^{(4)} (s_1^2 - s_3^2) + b_{ij}^{(5)} (s_2^2 - s_3^2) + b_{ij}^{(6)} s_1 s_2 + b_{ij}^{(7)} s_1 s_3 + b_{ij}^{(8)} s_2 s_3. \quad (62)$$

We need to determine all 9 constants $b_{ij}^{(n)}$ by constructing 9 equations. This is accomplished by choosing 9 combinations of $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = -(V_2^2 - m_j^2)$ in Eq. (52). This system of equations can be easily solved using a matrix inversion. Note that in principle we can generate an unlimited set of equations. This can be useful in a numerical application to obtain better numerical accuracy. We only need the full bubble residue in the case when the tadpole contribution is non-zero.

With the above prescription it is now easy to determine the spurious term for any value of the loop momentum. Finally we note that the integration over the term

$$\int dl \frac{\bar{b}_{ij}(l)}{d_i d_j} = b_{ij}^{(0)} \int dl \frac{1}{d_i d_j} = b_{ij} I_{ij}, \quad (63)$$

is now trivially done, giving us the bubble coefficient times the bubble master integral.

F. Construction of the tadpole coefficient

In the case there is a tadpole contribution we need to determine its coefficient from the relation

$$\bar{a}_i(l) = \text{Res}_i \left(\mathcal{A}_N(l) - \sum_{j \neq i} \frac{\bar{b}_{ij}(l)}{d_i d_j} - \frac{1}{2!} \sum_{j,k \neq i} \frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{3!} \sum_{j,k,l \neq i} \frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l} \right). \quad (64)$$

Using Eq. (18) the loop momentum can be decomposed in four orthonormal vectors

$$l^\mu = \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu + \alpha_4 n_4^\mu, \quad (65)$$

and

$$n_i \cdot n_j = \delta_{ij}; \quad g^{\mu\nu} = n_1^\mu n_1^\nu + n_2^\mu n_2^\nu + n_3^\mu n_3^\nu + n_4^\mu n_4^\nu. \quad (66)$$

The solution to the unitarity constraints ($d_i = 0$ with $q_i = 0$) gives an infinite set of solutions

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = m_i^2, \quad (67)$$

The residue of the amplitude becomes a tree-level amplitude with an implicit sum over the states of the cut particle

$$\text{Res}_i \left(\mathcal{A}_N(l^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}) \right) = \mathcal{M}^{(0)}(l_i^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}; p_{i+1}, \dots, p_i; -l_i^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}). \quad (68)$$

For example, the residue of the amplitude for the pure 6-gluon amplitude with d_5 factorizes into (see fig. 5)

$$\text{Res}_5 \left(\mathcal{A}_6(l^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}) \right) = \mathcal{M}_8^{(0)}(l_5^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}; p_6, p_1, p_2, p_3, p_4, p_5; -l_5^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}). \quad (69)$$

The maximum rank of the tadpole in the Standard Model is one, giving for the spurious term

$$\bar{a}_i = a_i^{(0)} + a_i^{(1)} s_1 + a_i^{(2)} s_2 + a_i^{(3)} s_3 + a_i^{(4)} s_4. \quad (70)$$

The coefficient a_i of the master integral I_i is easily obtained by e.g.

$$a_i^{(0)} = \frac{\bar{a}_i(l^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}) + \bar{a}_i(l^{\beta_1 \alpha_2 \alpha_3 \alpha_4})}{2}, \quad (71)$$

with $\alpha_1 = -\beta_1 = m_i$ and $\alpha_2 = \alpha_3 = \alpha_4 = 0$.

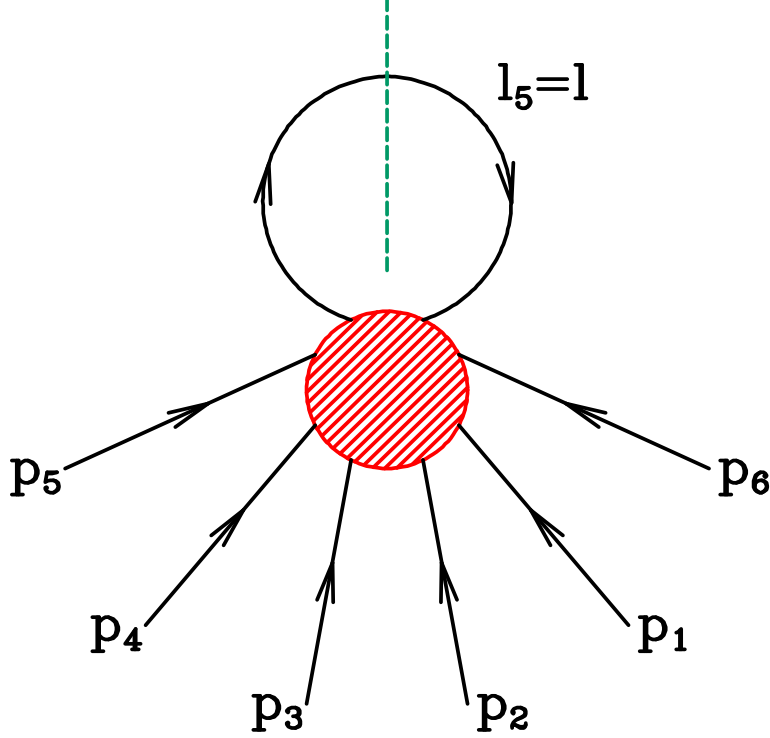


FIG. 5: The factorization of the 6-gluon amplitude for the calculation of the $\bar{a}_5(l)$ residue with the loop momentum parametrization choice $q_0 = -p_6 = p_1 + p_2 + p_3 + p_4 + p_5$.

III. NUMERICAL RESULTS

As an application we calculate the 4, 5 and 6 gluon scattering amplitudes at one-loop using the method of sec. 2. The cut-constructible parts of the ordered amplitudes are also known analytically ([34, 35, 36], [37], [4, 5, 38, 39, 40, 41, 42]), making a direct comparison possible. These multi-gluon scattering amplitudes form a good test for numerical procedures as they are the sum over a large number of Feynman graphs with significant gauge cancellations. Also, the 6-gluon amplitude was numerically evaluated using the integration-by-parts method [8]⁹. The numerical evaluation time using that method was around 9 seconds per ordered amplitude (on a 2.8GHz Pentium processor). We can compare this evaluation time with the unitarity method of the previous section. This directly compares the computational effort between a numerical method using Feynman diagrams with form factor expansion and the numerical unitarity method using analytical expressions for the tree-level amplitudes.

⁹ This Feynman diagram calculation yielded the full amplitude including rational terms.

To calculate the gluon scattering amplitude we need to determine all the master integral coefficients and combine these with the master integrals according to Eq. (3). It is straightforward to numerically evaluate a coefficient using the method outlined in the previous section. Given a set of external momenta we simply calculate the appropriate four vectors $\{v_i\}$ and $\{n_i\}$ according to Eqs. (28, 43, 45, 55, 57). From these vectors we construct the special loop momenta of Eqs. (26, 41, 53) which sets the appropriate denominators to zero. Using the analytically known leading order gluon amplitudes we can calculate the coefficient of Eqs. (38, 51, 63). Note that the multi-gluon scattering amplitudes do not get a contribution from the tadpole master integrals. We therefore need only to calculate the $b_{ij}^{(0)}$ -coefficient of the bubble residue and not the remaining 8 coefficients of the spurious term.

The first check on the numerical implementation is performed by calculating the ϵ^{-2} term, where $D = 4 - 2\epsilon$. This term gets contributions both from the box master integrals and from triangle master integrals with one leg off-shell. The ϵ^{-2} -term for a n -gluon ordered amplitude is proportional to the leading-order amplitude and is given by

$$m^{(1)}(1, 2, \dots, n) \sim -\frac{n}{\epsilon^2} \times m^{(0)}(1, 2, \dots, n) + \mathcal{O}(\epsilon^{-1}) . \quad (72)$$

The next check is the ϵ^{-1} term. Again, the term is proportional to the leading-order ordered amplitude. This term gets contributions from the one-, two- and three-leg off-shell box master integrals, one- and two-leg off-shell triangle master integrals and bubble master integrals. The contribution is given by

$$m^{(1)}(1, 2, \dots, n) \sim \left(-\frac{n}{\epsilon^2} + \frac{1}{\epsilon} \left(-\frac{11}{3} + \sum_{i=1}^n \log \left(\frac{s_{i,i+1}}{\mu^2} \right) \right) \right) \times m^{(0)}(1, 2, \dots, n) + \mathcal{O}(1) , \quad (73)$$

where $s_{i,i+1} = 2p_i \cdot p_{i+1}$ and $s_{n,n+1} = s_{n,1}$. As usual μ is the scale introduced to maintain the dimension of the integral in $4 - 2\epsilon$ dimensions.

When the implementation passes both non-trivial tests we have checked all coefficients proportional to infrared and collinear divergent master integrals. This leaves only the 3-leg off-shell triangle coefficients and the 4-leg off-shell box coefficients unchecked. Note that for the 4-gluon and 5-gluon ordered amplitudes there are no finite master integral contributions and hence all coefficients are checked by looking at the divergent parts. The 6-gluon amplitude gets a contribution from the finite 3-leg off-shell triangle master integral.

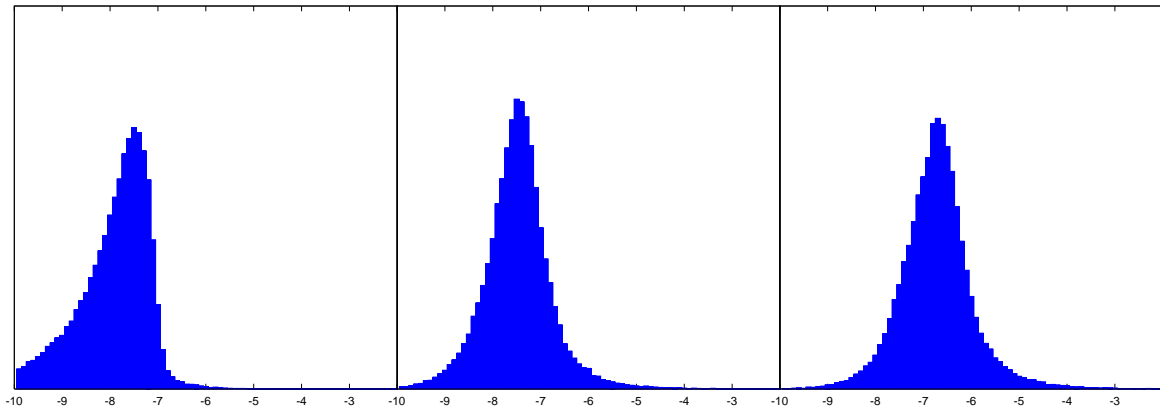


FIG. 6: The relative error for 100,000 ordered 4-gluon amplitude for the $(++--)$ helicity choice. The horizontal axis is the log-10 of the relative error of Eq. (75), the vertical axis is the number of events in arbitrary linear units. The left figure is the ϵ^{-2} contribution, the middle figure is the ϵ^{-1} contribution and the right figure is the finite part.

To compare with the analytic results we generate 100,000 flat phase space events for the $2 \rightarrow (n-2)$ gluon scattering using RAMBO [33]. The center-of-mass collision energy is \sqrt{S} . The events are required to have the following cuts on the outgoing gluons: a cut on the transverse energy, $E_T > 0.01 \times \sqrt{S}$, a maximum rapidity, $\eta < 3$ and a separation cut, $\Delta R > 0.4$.

The evaluation time for 10,000 events is: for a $2 \rightarrow 2$ gluon ordered helicity amplitude 9 seconds, for a $2 \rightarrow 3$ gluon ordered helicity amplitude 35 seconds and for a $2 \rightarrow 4$ gluon ordered helicity amplitude 107 seconds. Note that using the integration-by-parts method of ref. [8] the evaluation time for 10,000 events would be approximately 90,000 second. This means the unitarity method of section 2 improves the evaluation of the six-gluon amplitudes by a factor of approximately 900, almost 3 orders of magnitude. The six-gluon evaluation is only three times slower than the five gluon evaluation and eleven times slower than the four gluon amplitude. This can be understood by counting the number of coefficients needed to evaluate the scattering amplitude. The number of coefficients multiplying a non-zero master integral for the n -gluon scattering ordered amplitude is

$$\binom{n}{4} + \binom{n}{3} + \binom{n}{2} - n = \frac{n}{24} (n^3 - 2n^2 + 11n - 34) , \quad (74)$$

where the first term is the number of box coefficients, the second term the number of triangle coefficients and the third term gives the number of self energy coefficients. Finally, the

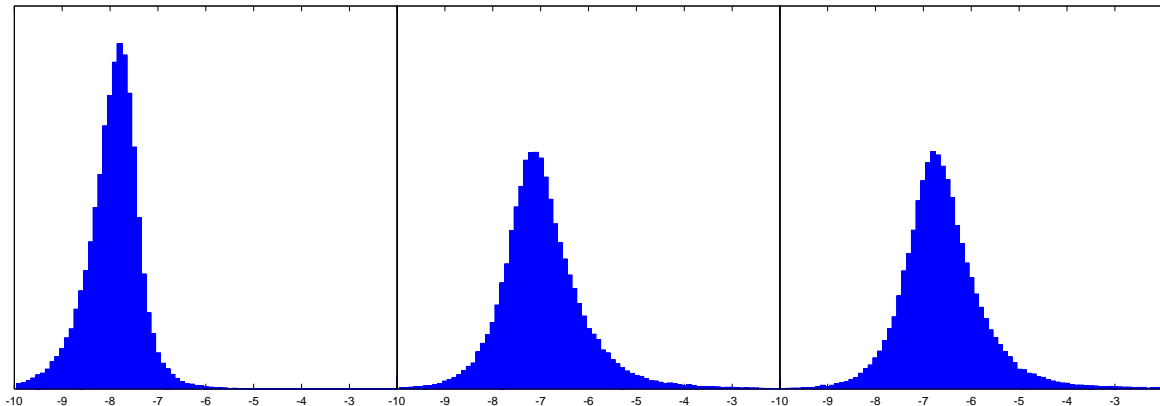


FIG. 7: Same as fig. 6, but the ordered 5-gluon amplitude for the $(++---)$ helicity choice.

last term subtracts the n external gluon bubble master integrals (because the corresponding master integral is zero). That is, the number of 4-gluon coefficients is six, the number of 5-gluon coefficients is twenty and the number of 6-gluon coefficients forty-four. The computational time roughly follows the number of coefficients to be calculated for the scattering. This is a very different scaling law than the $n!$ growth of a straightforward Feynman diagram expansion (as is the case for the integration-by-part algorithm of ref. [8]). The “factorization” of the diagrams into the tree-level blobs explains this large difference and causes the large gain in speed using the unitarity method compared to more conventional methods, replacing a $n!$ factorial growth by a n^4 power growth. The advantage of unitarity method with respect to a Feynman-diagram based approach becomes greater as the number of external particles grows (provided the tree-level amplitudes are known).

The numerical comparisons for the cut constructible part of $m_4(++--)$, $m_5(++---$) and $m_6(++-----)$ ordered helicity amplitudes are summarized in figs. (6,7,8)¹⁰. The degree of agreement is quantified by the expression

$$S = \log_{10} \left(\left| \frac{m_{\text{unitarity}}^{(1)} - m_{\text{analytic}}^{(1)}}{m_{\text{analytic}}^{(1)}} \right| \right). \quad (75)$$

In the 3 figures the 100,000 events are compared and binned in the quantity S for each of the 3 contributions: ϵ^{-2} , ϵ^{-1} and finite. As can be seen the majority of the events agree with a relative precision of 10^{-6} or better. This is more than sufficient for TEVATRON, LHC and

¹⁰ We also compared all other helicity combinations with the known results in the literature, leading to similar results.

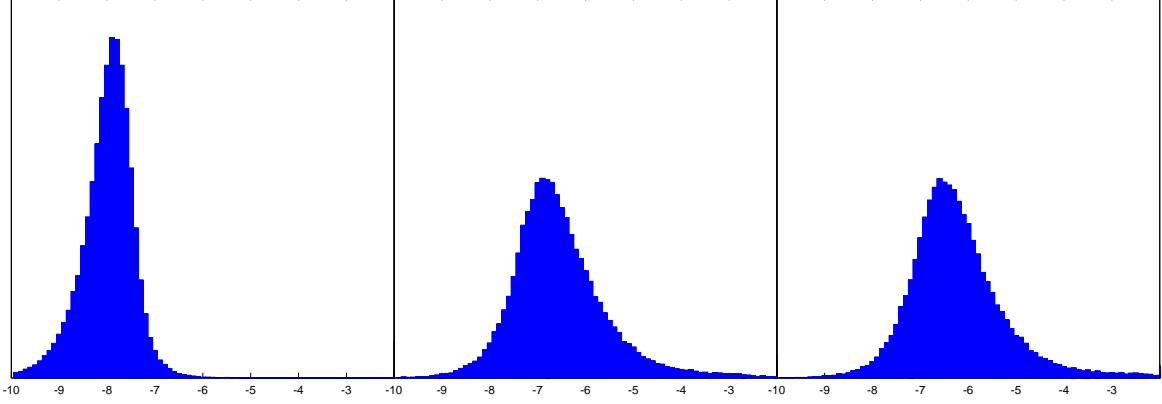


FIG. 8: Same as fig. 6, but the ordered 6-gluon amplitude for the $(+ + - - -)$ helicity choice.

ILC applications. However, for the ϵ^{-1} and finite parts a small portion of the events have a worse agreement. This is related to the fact that in these cases the amplitude receives a contribution from the bubble coefficient. In particular it is related to the calculation of the triangle subtraction term needed in the calculation of the bubble coefficient. For a small fraction of the phase space points there is a numerical instability in the matrix inversion needed to calculate the coefficients of the triangle spurious term in Eq. (50). To understand this better we simply rotate the two basis vectors of the trivial space of the triangle in the following manner

$$\begin{aligned} n_+^\mu &= n_1^\mu + i n_2^\mu \\ n_-^\mu &= n_1^\mu - i n_2^\mu, \end{aligned} \quad (76)$$

such that

$$n_+ \cdot n_+ = n_- \cdot n_- = 0; \quad n_+ \cdot n_- = 2. \quad (77)$$

Within this basis the spurious term becomes

$$\bar{c} = c'_0 + c'_1 s_+ + c'_2 s_+^2 + c'_3 s_+^3 + c'_4 s_- + c'_5 s_-^2 + c'_6 s_-^3, \quad (78)$$

where $s_+ = l \cdot n_+$ and $s_- = l \cdot n_-$. By choosing 7 loop momenta $\{l_i\}_{i=1}^7$ we get the set of equations

$$m_i = \sum_{j=1}^7 A_{ij} c'_j \Rightarrow c'_j = \sum_{i=1}^7 m_i A_{ij}^{-1}, \quad (79)$$

where $m_i = \bar{c}(l_i)$ and $A_{ij} = (1, s_+(l_i), s_+^2(l_i), s_+^3(l_i), s_-(l_i), s_-^2(l_i), s_-^3(l_i))$. We note that the

matrix A_{ij} is a double vanderMonde-matrix [43], i.e. a matrix of the form

$$A_{ij} = \begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 & \beta_1 & \beta_1^2 & \beta_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 & \beta_2 & \beta_2^2 & \beta_2^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_7 & \alpha_7^2 & \alpha_7^3 & \beta_7 & \beta_7^2 & \beta_7^3 \end{pmatrix}, \quad (80)$$

which is known to give an unstable inverse. No numerical procedure is known to stabilize the calculation of the inverse. In the figures we simply used the inverse anyway to see if these numerical issues would appear. We see that for a small fraction of the events they indeed appear in the case of the ϵ^{-1} and finite contributions. On the other hand it is important to realize we have an infinite set of equations (i.e. an infinite set of loop momenta fulfilling the triple cut unitarity condition) with only 7 coefficients to determine. This means we can achieve arbitrary precision using a χ^2 -type fitting procedure, at the cost of more computer time. We did not pursue this method for this paper.

The final issue is the presence of Gram determinants in the box, triangle and bubble coefficients. We need to be aware of the Gram determinants when performing phase space integrals. We identify from the procedure in the last section two separate mechanisms of generating Gram determinant type of denominator factors in the coefficient.

The first mechanism is straightforward. The solutions of the unitarity constraints for the box by quadruple cut gives a loop momentum which is proportional to the inverse of the square root of the 3-particle Gram determinant as can be seen in Eqs. (25-27)

$$\frac{\delta_{k_1 k_2 k_3}^{k_1 k_2 \mu}}{\Delta(k_1, k_2, k_3)} \sim \frac{1}{\sqrt{\Delta(k_1, k_2, k_3)}}. \quad (81)$$

Because the maximum rank of the box tensor integral is 4, we can get at most terms of order $\Delta(k_1, k_2, k_3)^{-2}$. More precisely

$$d_{ijkl} \equiv A + \frac{B}{\Delta(k_1, k_2, k_3)^{1/2}} + \frac{C}{\Delta(k_1, k_2, k_3)} + \frac{D}{\Delta(k_1, k_2, k_3)^{3/2}} + \frac{E}{\Delta(k_1, k_2, k_3)^2}, \quad (82)$$

with $k_1 = (q_j - q_i)$, $k_2 = (q_k - q_j)$ and $k_3 = (q_l - q_k)$. Similar, for the triangle and bubble coefficients we get respectively up to $\Delta(k_1, k_2)^{-3/2}$ and $\Delta(k_1)^{-1}$ terms.

The second source of Gram determinants is more subtle and is generated for 5-particle or higher scattering amplitudes. It can happen that e.g. $d_5(l_{1234}) \rightarrow 0$ causing an instability at a specific phase space point. It should be treated with care when performing a numerical phase space integration.

IV. SUMMARY AND OUTLOOK

We have presented a formulation of the 4-dimensional unitarity cut method in a physical language using the van Neerven-Vermaseren basis of ref. [30]. This basis is also used to reduce 5- (or higher) point tensor integrals to 4-point tensor integrals. When applied to 4- (or lower) point tensor integrals the decomposition of the loop momentum generates extra basis vectors spanning a “trivial” space. This trivial space is orthogonal to the external momenta of the loop integral. When applying unitarity cuts, the loop momentum dependence in the trivial space generate so-called spurious terms [15]. These spurious terms integrate to zero when considering an individual cut diagram. However, when combining the double, triple and quadruple cuts care has to be taken not to double count. These spurious terms play an important role in the subtraction schemes needed to avoid the double counting problem. The procedure outlined in this paper allows us to solve the unitarity constraints without resorting to the explicit 4-dimensional spinor formalisms used in analytic calculations. This makes the method equally applicable to processes with (complex) masses for the internal lines.

When applying the 4-dimensional unitarity cuts to the amplitude, both the master integral coefficients and the spurious terms are calculable in terms of factorized products of tree-level amplitudes. The cut lines cause two of the external momenta of the tree-level amplitude to be complex. Existing leading-order generators, such as VECBOS [44] and NJETS [45, 46], can be upgraded to allow for two complex external momenta without much effort. Once these upgraded tree-level generators are interfaced to the numerical program based on method described in this paper, they can be converted to one-loop generators without any additional calculations. This will allow us to automate the calculation of the cut-constructible part of the one-loop amplitudes for many important background processes at the TEVATRON and LHC such as $PP \rightarrow 4$ or 5 jets, $PP \rightarrow W + 3$, or 4 jets, $PP \rightarrow t + \bar{t} + 1, 2$ or 3 jets, $PP \rightarrow t + \bar{t} + b + \bar{b} + 0$ or 1 jets (massless bottom quark) etc..

As an example, we took the analytically known 4-, 5- and 6-gluon ordered tree-level helicity amplitudes and used them to evaluate the cut-constructible part of the 4-, 5- and 6-gluon ordered one-loop helicity amplitudes. Especially the 6-gluon amplitudes are very complex, demonstrating the power of the method in the paper. The computational time required to evaluate the amplitudes is fast enough for serious TEVATRON, LHC and ILC

applications. The numerical instabilities in certain phase space points due to additional linear dependences between the particle momenta (the “Gram determinant instabilities”) are easily identified in the master integral coefficients. The degree of the instabilities are the same as one gets from analytic unitarity calculations.

The ultimate goal is to construct a NLO parton level generator. To reach that point, two outstanding challenges remain. The first one is the completion of the scattering amplitude, i.e. an automated calculation of the rational part. The second, far more difficult, challenge are the phase space integrations of both the one-loop amplitudes and the bremsstrahlung contributions.

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